ADVANCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2020-2021 (MCB, MCE, MEG, MPC, MPG, PCM, PEM)

SECTION A: Attempt all questions. (55 marks)

1) **Answer:**

$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \frac{\sin(4-4)}{4-4} = \frac{0}{0}\ IF$

By hospital rule:

$\lim_{x\to 4}\frac{\sin(x-4)}{x-4} = \lim_{x\to 4}\frac{[\sin(x-4)]'}{(x-4)'} = \lim_{x\to 4}\frac{\cos(x-4)}{1} = \cos 0 = 1$

2) **Answer:**

$w = \sqrt{3} - i \Rightarrow a = \sqrt{3}, b = -1$

Let ρ and φ be the modulus and argument of the complex number w.

$\rho = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$: The modulus

$\cos\varphi = \frac{a}{\rho} = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \begin{cases}\frac{\pi}{6}\\ \frac{11\pi}{6}\end{cases}$ $\quad \sin\varphi = \frac{b}{\rho} = -\frac{1}{2} \Rightarrow \varphi' = \frac{\pi}{6} \Leftrightarrow \varphi = \begin{cases}\frac{11\pi}{6}\\ \frac{7\pi}{6}\end{cases}$

The argument is $\frac{11\pi}{6}$

The required trigonometric form is $w = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

3) **Answer:**

Let the curve $\psi(x,y) \equiv 2x - 5x^3 = 1 + xy$ .... (1)

In polar coordinates, we know that: $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi\end{cases}$ .... (2)

Putting (2) into (1) we get:

$2\rho\cos\varphi - 5\rho^3\cos^3\varphi = 1 + \rho^2\sin\varphi\cos\varphi$

The required polar form is:

$\psi(\rho,\varphi) \equiv 2\rho\cos\varphi - 5\rho^3\cos^3\varphi = 1 + \frac{1}{2}\rho^2\sin 2\varphi$: *The required form!*

Or

$\psi(\rho,\varphi) \equiv \rho(2\cos\varphi - 5\rho^2\cos^3\varphi - \frac{1}{2}\rho\sin 2\varphi = 1$: *The required form!*

4) **Answer:**

$\ln(e^x) = \ln(e^3) + \ln(e^5) \Rightarrow x = 3 + 5 \Leftrightarrow x = 8$

$S = \{x\} = \{8\}$

5) **Answer:**

$x^2.2^x - 2^x = 0 \quad 2^x \neq 0$

$2^x(x^2-1) = 0 \Leftrightarrow x^2 - 1 = 0$

$x^2 = 1 \Rightarrow x = \pm 1$

$S = \{x_1, x_2\} = \{-1, 1\}$

6) **Answer:**

$e^{\frac{dy}{dx}} = x + 1$

$\ln\left(e^{\frac{dy}{dx}}\right) = \ln(x+1)$

$\frac{dy}{dx} = \ln(x+1)$

$dy = \ln(x+1)dx$

y = $\int \ln(x+1)dx$

Evaluating an integral:

$\int \ln(x+1)\, dx$

Method1: integration by parts

Let u = ln(x+1) ⇒ du = $\frac{dx}{x+1}$, $dv = dx \Rightarrow v = x$

$\int u dv = uv - \int v du$

$\int \ln(x+1)dx = x\ln(x+1) - \int \frac{x}{x+1}dx$

= xln(x+1) - $\int \frac{x+1-1}{x+1}dx$

= x ln(x+1) - $\int dx + \int \frac{dx}{x+1}$

= x ln(x+1) – x + ln(x+1) + C

= (x+1)ln(x+1) – x + C

y(x) = (x+1) ln(x+1) – x + C

x = 0, y = 3 ⇒ C = 3

The required solution is:

y(x) = (x+1) ln(x+1) – x + 3

Method 2: integration by substitution

$\int \ln(x+1)dx$

Let u = x+1 ⇒ dx = du

Hint: $\int \ln u\, du = u \ln u - u + C$

$\int \ln(x+1)dx = \int \ln u\, du = u \ln u - u + C$

= (x+1) ln(x+1) – x – 1+ C

= (x+1) ln(x+1) – x + C

y (x) = (x+1) ln (x+1) – x + C

x = 0, y = 3 ⇒ $C = 3$

The required solution is:

y (x) = (x+1) ln (x+1) – x + 3

7) Answer:

A (2, -1, 3), P ≡ x – 2y + 3z – 1 = 0, let find the plane $\pi \parallel P$ *and* $A \in \pi$.

Since the two planes are parallel, they have the same normal vector $\vec{n}(1,-2,3)$. Let π passes through an arbitrary point A'(x, y, z) whose position vector $\vec{r}(x,y,z)$ *and A with its position vector* $\vec{r}_A = (2,-1,3)$ *and*, it follows that:

$\pi \equiv \vec{n}(\vec{r} - \vec{r}_A) = 0$

$\pi \equiv (x-2) - 2(y+1) + 3(z-3) = 0$

$\pi \equiv x - 2y + 3z - 13 = 0$: *Required plane!*

8) Answer:

$z^2 = 1 + i$

$z = \sqrt{1+i} = x + yi$

Hint: $x = \pm\frac{\sqrt{2a+2\sqrt{a^2+b^2}}}{2}$ $\qquad y = \pm\frac{\sqrt{-2a+2\sqrt{a^2+b^2}}}{2}$

$b = 1 > 0$: *x and y have the same sign.*

$x = \pm\frac{\sqrt{2a+2\sqrt{a^2+b^2}}}{2} = \pm\frac{\sqrt{2+2\sqrt{2}}}{2}$ $\qquad y = \pm\frac{\sqrt{-2a+2\sqrt{a^2+b^2}}}{2} = \pm\frac{\sqrt{-2+2\sqrt{2}}}{2}$

$z_1 = -\frac{\sqrt{2+2\sqrt{2}}}{2} - \frac{\sqrt{-2+2\sqrt{2}}}{2}i$ $\qquad z_2 = \frac{\sqrt{2+2\sqrt{2}}}{2} + \frac{\sqrt{-2+2\sqrt{2}}}{2}i$

$S = \{z_1, z_2\} = \left\{-\frac{\sqrt{2+2\sqrt{2}}}{2} - \frac{\sqrt{-2+2\sqrt{2}}}{2}i, \frac{\sqrt{2+2\sqrt{2}}}{2} + \frac{\sqrt{-2+2\sqrt{2}}}{2}i\right\}$

9) **Answer:**

a) $y = x^2 + 6x + 5$

$= (x^2 + 6x + 9 - 4)$

$= (x+3)^2 - 4$ vertex form

The vertex is: (-3, -4) or $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) = (-3, -4)$

b) Axis of symmetry: $l \equiv x = -3$

10) **Answer:**

$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^3 x}$

Method 1: Use of change of variables

Let $u = \tan x \Longrightarrow dx = \frac{du}{\sec^2 x} = \frac{du}{\sec^2 x} = \frac{du}{1+\tan^2 x} = \frac{du}{1+u^2}$

Since $u = \tan x$: $\begin{cases} x \to \frac{\pi}{2} \Rightarrow u \longrightarrow +\infty \\ x \to 0 \Rightarrow u \longrightarrow 0 \end{cases}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^3 x} = \int_0^{+\infty} \frac{du}{(1+u^3)(1+u^2)} \Longrightarrow$ *Improper integral!*

Using partial fraction:

$\frac{1}{(1+u^3)(1+u^2)} = \frac{A}{1+u} + \frac{Bu+C}{u^2-u+1} + \frac{Du+E}{1+u^2}$

$= \frac{A(u^2-u+1)(u^2+1)+(Bu+C)(1+u)(u^2+1)+(DU+E)(1+u^3)}{(1+u^3)(1+u^2)}$

$$\Rightarrow \begin{cases} A+B+D=0 \\ -A+B+C+E=0 \\ 2A+B+C=0 \\ -A+B+C+D=0 \\ A+B+E=1 \end{cases} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \end{array}\right]$$

From the row reduced echelon matrix we get:

$A = \frac{1}{6}, B = -\frac{2}{3}, C = \frac{1}{3}, D = \frac{1}{2}, E = \frac{1}{2}$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\int_0^{+\infty}\frac{du}{(1+u)(u^2-u+1)(u^2+1)}$$

$$\int\frac{du}{(1+u)(u^2-u+1)(u^2+1)}=\frac{1}{6}\int\frac{du}{1+u}-\frac{1}{3}\int\frac{2u-1}{u^2-u+1}du+\frac{1}{2}\int\frac{u+1}{u^2+1}du$$

$$=\frac{1}{6}ln(1+u)-\frac{1}{3}\int\frac{2}{u^2-u+1}\frac{d(u^2-u+1)}{2u-1}+\frac{1}{4}\int\frac{2}{u^2}\frac{d(u^2+1)}{2u}+\frac{1}{2}\int\frac{d}{u^2+1}$$

$$=\frac{1}{6}ln|1+u|-\frac{1}{3}ln|u^2-u+1|+\frac{1}{4}ln|u^2+1|+\frac{1}{2}arctan\,u+C$$

$$=\frac{1}{12}\left[ln\left|\frac{(1+u)^2(u^2+1)^3}{(u^2-u+1)^4}\right|\right]+\frac{1}{2}arctan\,u+C$$

$$\int_0^{+\infty}\frac{du}{(1+u)(u^2-u+1)(u^2+1)}=\lim_{b\to+\infty}\left[\frac{1}{12}ln\left|\frac{(1+b)^2(b^2+1)^3}{(b^2-b+1)^4}\right|+\frac{1}{2}\,arctan\,b\right]$$

$$=\frac{1}{12}ln\left(\lim_{b\to+\infty}\frac{b^8}{b^8}\right)+\frac{1}{2}arctan+\infty$$

$$=\frac{1}{2}x\frac{\pi}{2}=\frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\frac{\pi}{4}$$

Method 2: Use of special definite integrals properly $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ ......(1)

As we have trigonometric functions and a = $\frac{\pi}{2}$, it follows that complementary arc formulae will help us:

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\frac{sin^3x}{cos^3x}}=\int_0^{\frac{\pi}{2}}\frac{cos^3x}{cos^3x+sin^3x}dx$$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\int_0^{\frac{\pi}{2}}\frac{cos^3x}{cos^3x+sin^3x}dx \quad ....\ (2)$$

From (1) into (2):

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\int_0^{\frac{\pi}{2}}\frac{cos^3\left(\frac{\pi}{2}-x\right)}{cos^3\left(\frac{\pi}{2}-x\right)+sin^3\left(\frac{\pi}{2}-x\right)}dx=\int_0^{\frac{\pi}{2}}\frac{cos^3x}{cos^3x+sin^3x}dx \quad ....(3)$$

Adding (2) and (3), we get:

$$2\int_0^{\frac{\pi}{2}}\frac{dx}{1+tan^3x}=\int_0^{\frac{\pi}{2}}\frac{cos^3x+sin^3x}{cos^3x+sin^3x}dx=\int_0^{\frac{\pi}{2}}dx=x|_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

11) **Answer:**

y = x^2 + 1 and y = x^3

Formula:

$$A=\int_a^{b}\left(y_1(x)-y_2(x)\right)dx$$

The required area is given by:

$$A=\left|\int_{-1}^{1}(x^3-x^2-1)dx=\left|\frac{x^4}{4}-\frac{x^3}{3}-x\right|_{-1}^{1}\right|=\left|\frac{1}{4}-\frac{1}{3}-1-\frac{1}{4}-\frac{1}{3}-1\right|=\frac{8}{3}\ units\ area$$

12) **Answer:**

P_o = 100,000, r = 0.05, t = 40

Formula: P = P_oe^{rt}

P = 100,000$e^{0.05x40}$ = 100,000e^2 ≈ $ 738,906

13) **Answer:**

f(x) = ln(x+cosx)

Method 1: Use of logarithmic functions differentiation property

f(x) = ln(u(x)) ⇒ f'(x) = $\frac{u'(x)}{u(x)}$

f'(x) = $\frac{(x+cosx)'}{x+cosx} = \frac{1-sinx}{x+cosx}$

Method 2: Use of Leibniz formula or chain differentiation.

Let y = f(x) ⇒ y = ln(x+cosx)

u = x+cosx ⇒ y = lnu

$\frac{du}{dx} = 1 - sinx \quad \frac{dy}{du} = \frac{1}{x+\cos x}$

$\frac{dy}{dx} = \frac{dy}{du} x \frac{du}{dx} = \frac{1-sinx}{x+cosx}$

14) **Answer:**

Linearization of sin x cos² x

Method 1: Use basic linearization formulae

sinx cos² x = $sinx\left(\frac{1+cos2x}{2}\right) = \frac{1}{2}(sinx + sinxcos\ 2x)$

$= \frac{1}{2}(sinx + \frac{1}{2}(sin3x - sinx)$

$= \frac{1}{2}\left(\frac{1}{2}sinx + \frac{1}{2}sin3x\right)$

$= \frac{1}{4}sinx + \frac{1}{4}sin3x$

Method 2: Use of complex numbers

sinx = $\frac{e^{ix}-e^{-ix}}{2i}$ cosx = $\frac{e^{ix}+e^{-ix}}{2}$

sinxcos²x = $\frac{e^{ix}-e^{-ix}}{2i}.\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$

$= \frac{1}{8i}(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})^2$

$= \frac{1}{8i}(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})(e^{ix} + e^{-ix})$

$= \frac{1}{8i}(e^{2ix} - e^{-2ix})(e^{ix} + e^{-ix})$

$= \frac{1}{8i}(e^{3ix} + e^{ix} - e^{-ix} - e^{-3ix})$

$= \frac{1}{4}\left(\frac{e^{ix}-e^{-ix}}{2i} + \frac{e^{3ix}-e^{-3ix}}{2i}\right)$

$= \frac{1}{4}sinx + \frac{1}{4}sin3x$

15) **Answer:**

y = x² – 1

The interval of integration is found such that y = x² – 1 and y = 0 then, x² – 1 = 0 ⇒ x = ±1

Formula: V = $\pi\int_a^b f^2(x)dx$

V = $\pi\int_{-1}^{1}(x^2-1)^2dx = \pi\int_{-1}^{1}x^4 - 2x^2 + 1dx$

V = $2\pi\left(\frac{x^5}{5} - \frac{2x^3}{3} + x|_0^1\right) = 2\pi\left(\frac{1}{5} - \frac{2}{3} + 1\right) = 2\pi\frac{3-10+15}{15} = \frac{16}{15}\pi$ units of volume

SECTION B: CHOOSE ANY THREE QUESTIONS (45 marks)

16) Answer:

a) $\int tan^3xdx = \int tanx.tan^2x\,dx = \int tanx.(sec^2x-1)\,dx$

$= \int tanxsec^2xdx \int tanx\,dx$

Taking $I_1 = \int tanxsec^2xdx$ and $I_2 = \int tanx\,dx$

$I_1 = \int tanxsec^2xdx$

Let u = tan x $\Rightarrow$ dx = $\frac{du}{sec^2x}$

$I_1 = I_1 = \int tanxsec^2xdx = \int u.sec^2x\frac{du}{sec^2x} = \frac{u^2}{2} + C_1 = \frac{tan^2x}{2} + C_1$

$I_2 = \int tanxdx = \int \frac{sinx}{cosx}dx$:

Let $u = cosx \Rightarrow dx = -\frac{du}{sinx}$

$I_2 = \int tanx\,dx = \int \frac{sinx}{cosx}dx = -\int \frac{sinx}{u}.\frac{du}{sinx} = -ln|u| + C_2 = -ln|cosx| + C_2$

$\therefore \int tan^3xdx = \frac{tan^2x}{2} + ln|cosx| + C$

b) y''+ 8y'+25y = 0, x_o = 0, y_o = 2, and y_o = 1

CE $\equiv \lambda^2 + 8\lambda + 25 = 0$

$\Delta = 64 - 100 = -36 < 0$

$\lambda_{1,2} = \frac{-8\pm6i}{2} = \begin{cases}\lambda_1 = -4+3i\\ \lambda_2 = -4-3i\end{cases} \Rightarrow \alpha = -4, \beta = 3 \Rightarrow y(x) = (C_1cos\beta x + C_2sin\beta x)e^{\alpha x}$

$y(x) = (C_1cos3x + C_2sin3x)e^{-4x}$

$y'^{(x)} = (-3C_1cos3x + 3C_2sin3x)e^{-4x} - 4(C_1cos3x + C_2sin3x)e^{-4x}$

$y(0) = 2, y'(0) = 1 \Rightarrow \begin{cases} C_1 = 2\\ -4C_1 + 3C_2 = 1\end{cases} \Leftrightarrow C_1 = 2, C_2 = 3$

$\therefore y(x) = (2cos3x + 3sin3x)e^{-4x}$

17) Answer:

a) $P_1 = 10000e^{kt}$, $P_2 = 20000e^{0.01t}$, t_o = 2000 t = 2040 – 2000 = 40years

$P_1(40) = P_2(40) \Rightarrow 10000e^{40k} = 20000e^{0.01x40}$

$e^{40k} = 2e^{0.4}$

$ln(e^{40k}) = ln(2e^{0.4})$

$40k = ln\,2 + ln(e^{0.4})$

$40k = ln2 = 0.4$

$k = \frac{ln2+0.4}{40} \approx 0.027$

b) $\alpha \ni P, P\,(2,-3,4)$

The normal vector of the required α is collinear to the vector $\overrightarrow{ab} = k\vec{n}, k \in \mathbb{R}$:

$\overrightarrow{ab} = (-3,-3,-4) \Rightarrow \vec{n}$ = 3, 3, 4)

The equation of plane α is given by:

$\alpha \equiv A(x-x_o) + B(y-y_o) + C(z-z_o) = 0$ $\quad P\,(x_o, y_o, z_o)$

$\alpha \equiv 3(x-2) + 3(y+3) + 4(z-4) = 0$

$\equiv 3x + 3y + 4z - 6 + 9 - 16 = 0$

$\equiv 3x + 3y + 4z - 13 = 0$

$\alpha \equiv 3x + 3y + 4z - 13 = 0$: Required equation of plane!

c) Let $\vec{u} = \vec{i} + 2\vec{j} - 3\vec{k}, \vec{v} = 3\vec{i} + \lambda\vec{j} + \vec{k}, \vec{w} = \vec{i} + 2\vec{j} + 3\vec{k}$

The set of three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$, *is coplanar iff*:

det $(\vec{u}, \vec{v}, \vec{w})=0 \Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 2 & \lambda & 2 \\ -3 & 1 & 3 \end{vmatrix} = 0 \Rightarrow 6\lambda - 36 = 0 \Leftrightarrow \lambda = 6$

Therefore, the three given vectors are coplanar iff $\lambda = 6$

18) Answer:

a) $l_{\overline{OA}} \equiv \begin{cases} x = 3 + 3\lambda \\ y = 2 + 2\lambda \end{cases}$ Or $l_{\overline{OA}} \equiv \begin{cases} x = 3\lambda \\ y = 2\lambda \end{cases}$

$\frac{x-3}{3} = \frac{y-2}{2}$ or $\frac{x}{3} = \frac{7}{2}$

2x-3y =0

y = $\frac{2}{3}x$

$l_{\overline{AB}} \equiv \begin{cases} x = 3 + 2\lambda \\ y = 2 - 2\lambda \end{cases}$ Or $l_{\overline{AB}} \equiv \begin{cases} x = 5 + 2\lambda \\ y = 0 - 2\lambda \end{cases}$

$\frac{x-3}{2} = \frac{y-2}{-2}$ or $\frac{x-5}{2} = \frac{y}{-2}$

x+y -5 =0

y = -x+5

b) The required area is bounded by: y = $\frac{2}{3}x, y = -x + 5$ *and* $x - axis$ $I = [0.5]$

$A_1 = \int_0^3 \frac{2}{3}x dx = \frac{2}{3}\frac{x^2}{2}|_0^3 = \frac{2}{3}x\frac{9}{3} = 3cm^2$

$A_2 = \int_3^5 (-x + 5)dx = (\frac{x^2}{2} + 5x)|_3^5 = \frac{25}{2} + 25 + \frac{9}{2} - 15 = 2cm^2$

The required area is $A = A_1 + A_2 = 3cm^2 + 2cm^2 = 5cm^2$

c) Area = $\frac{BasexHeight}{2} = \frac{5x2}{2}cm^2 = 5\ cm^2$

d) f is not specified, but we analogically have two straight lines y = $\frac{2}{3}x, y = -x + 5$ representing two different functions. For each of functions we find the average value as follows:

$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

For $y = \frac{2}{3}x$:

$f_{av} = \frac{1}{5}\int_0^5 \frac{2}{3}xdx = \frac{1}{5}x\frac{2}{3}|_0^5 = \frac{25}{15} = \frac{5}{3}$

For $y = -x + 5$:

$f_{av} = \frac{1}{5}\int_0^5 (-x + 5)dx = \frac{1}{5}\left(-\frac{x^2}{2} + 5x\right)|_0^5 = \frac{1}{5}\left(\frac{25}{2} + 15\right) = \frac{5}{2}$

19) Answer:

	Speeding violations in last year	No speeding violation in last year	Total
Car phone use	25	280	305
No car phone use	45	405	450

Total	70	685	755

Let A: a person is a car phone user n(A) = 305

B: a person had no violation in last year n(B) = 685

C: a person had violation in last year n(C) = 70

D: a person is not a car phone user n(D) = 450

a) P(A) = $\frac{305}{755} = \frac{61}{151}$

b) P(B) = $\frac{685}{755} = \frac{137}{151}$

c) P(A∩B) = $\frac{280}{755} = \frac{56}{151}$

d) $P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{305}{755} + \frac{685}{755} + \frac{280}{755} = \frac{710}{755}\frac{142}{151}$

e) P(A\C)= $\frac{P(A\cap C)}{P(C)} = \frac{25/755}{70/755} = \frac{25}{70} = \frac{5}{14}$

f) P(B\D) = $\frac{P(B\cap D)}{P(D)} = \frac{405/755}{450/755} = \frac{405}{450} = \frac{9}{10}$

20) Answer:

$F_n(x) = \frac{x^n}{1+x^2}, x \in \mathbb{R}$: $I_n = \int_0^1 F_n(x)dx,$

a) $I_1 = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$ let $u = 1 + x^2 \Rightarrow dx = \frac{du}{2x}$

$\frac{1}{2}\int_0^1 \frac{du}{2x} = \frac{1}{2}\, x\, ln|1 + x^2||_0^1 = \frac{1}{2}ln2$

b) $I_1 + I_3 = \int_0^1 \frac{x^3+x}{1+x^2}dx = \int_0^1 xdx = \frac{1}{2}\, x^2|_0^1 = \frac{1}{2}$

$I_3 = \frac{1}{2} - I_1 = \frac{1}{2} - \frac{1}{2}ln2 = \frac{1}{2}(1 - ln2)$

c) $I_{2p} + I_{2p+2} = \int_0^1 \left[\frac{x^{2p}}{1+x^2} + \frac{x^{2p+2}}{1+x^2}\right]dx$

$= \int_0^1 \left[\frac{x^{2p}+x^{2p}x^2}{1+x^2}\right]dx$

$= \int_0^1 \left[\frac{x^{2p}(1+x^2)}{1+x^2}\right]dx$

$= \int_0^1 x^{2p}dx = \frac{x^{2p+1}}{2p+1}\Big|_0^1 = \frac{1}{2p+1}$ *Proved!*

d) $I_2 = \int_0^1 \frac{x^2}{1+x^2}dx = \int_0^1 \frac{1+x^2-1}{1+x^2}dx = \int_0^1 dx - \int_0^1 \frac{dx}{1+x^2} = x|_0^1 - arctan\, x|_0^1 = 1 - \frac{\pi}{4}$

From c), we know that $I_{2p} + I_{2p+2} = \frac{1}{2p+1} \Rightarrow I_{2p+2} = \frac{1}{2p+1} - I_{2p}$

$I_4 = I_{2.1+2} = \frac{1}{3} - I_2 = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{4-12+3\pi}{12} = \frac{3\pi-8}{12}$

$I_6 = I_{2.2+2} = \frac{1}{5} - I_4 = \frac{1}{5}\frac{3\pi-8}{12} = \frac{12-15\pi+40}{60} = \frac{52-15\pi}{60}$